Centre Number

Student Number

SCEGGS Darlinghurst

2009
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\lim_{x \to 0} \frac{\sin 3x}{5x}$ **1**

(b) Find the co-ordinates of the point *P* which divides *AB* externally in the ratio $2:3$ where $A(1, -4)$ and $B(6, 9)$. **2**

(c) Solve for *x*: **3**

$$\frac{4}{x-1} \geq 1$$

(d) The angle between two lines $y = mx$ and $y = \frac{1}{3}x$ is $\frac{\pi}{4}$. **2**

Find the exact values of *m*.

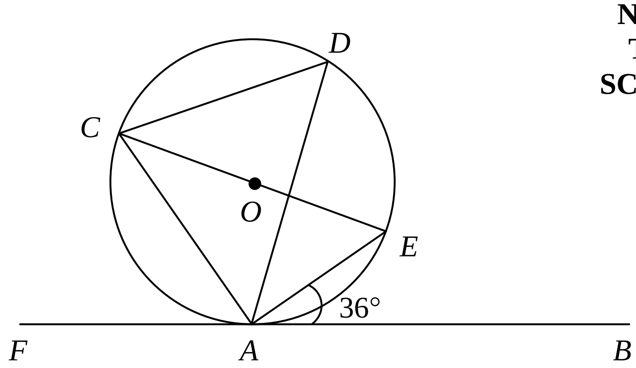
(e) If α, β and γ are the roots of $x^3 - 3x + 5 = 0$ find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **2**

(f) Use the table of standard integrals to find $\int \sec 2x \tan 2x \, dx$. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a)



NOT TO SCALE

FB is a tangent meeting a circle at *A*. *CE* is the diameter, *O* is the centre and *D* lies on the circumference. $\angle BAE = 36°$.

(i) Find the size of $\angle ACE$, giving reasons. **1**

(ii) Find the size of $\angle ADC$, giving reasons. **2**

(b) Find $\int \frac{dx}{\sqrt{25 - 4x^2}}$ **2**

(c) (i) If $\sin x - \sqrt{3}\cos x = R\sin\left(x - \alpha\right)$ find R and α where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Find the general solution for $\sin x - \sqrt{3}\cos x = \sqrt{2}$ (leave your answer in exact form). **2**

(d) Colour blindness effects 5% of all men. What is the probability that any random sample of 20 men should contain:

(i) no colour blindness. **1**

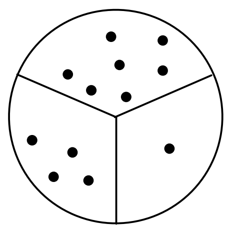
(ii) two or more colour blind men (to 3 decimal places). **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\frac{3}{2}} \sqrt{9 - x^2}\ dx$ using the substitution $x = 3\sin\theta$. **3**

(b) Twelve points lie inside a circle. No three points are collinear. Seven of the points lie in sector 1, four lie in sector 2 and the other point lies in sector 3.



(i) Show that 220 triangles can be made using these points. **1**

(ii) One triangle is chosen at random from all possible triangles. Find the probability that the triangle chosen has one vertex in each sector. **1**

(iii) Find the probability that the vertices of the triangle chosen all lie in the same sector. **1**

(c) (i) Sketch the graph of the function $f(x) = e^x - 2$. **1**

(ii) On the same diagram sketch the graph of the inverse function $f^{-1}(x)$. **1**

(iii) State the equation of the function $f^{-1}(x)$. **1**

(iv) Explain why the x co-ordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 2 = 0$. **1**

(v) One root of the equation $e^x - x - 2 = 0$ lies between $x = 1$ and $x = 2$. Use one application of Newton's method, with a starting value of $x = 1.5$, to approximate the root, to 2 decimal places. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

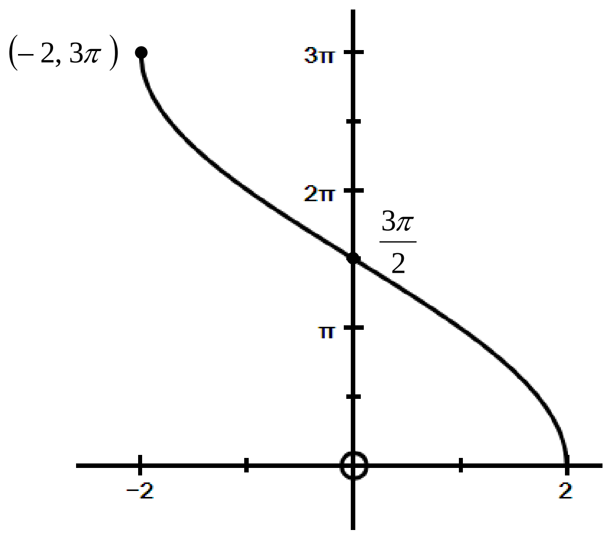
(a) $P\left(4p, 2p^2\right)$ is a variable point on the parabola $x^2 = 8y$.
The normal at *P* cuts the *y*-axis at *A* and *R* is the midpoint of *AP*.

(i) Show that the normal at *P* has equation $x + py = 4p + 2p^3$. **2**

(ii) Show that *R* has co-ordinates $\left(2p, 2p^2 + 2\right)$. **2**

(iii) Show that the locus of *R* is a parabola and find its vertex and focus. **3**

(b) The graph of $y = a\cos^{-1} bx$ is drawn below.



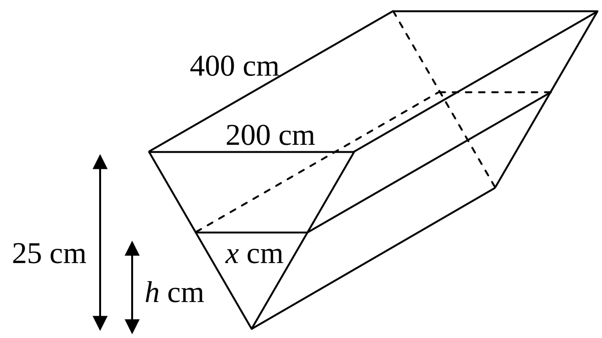
(i) Find *a* and *b*. **2**

(ii) Find the exact area bound by the curve and the *y*-axis for $0 \le y \le \dfrac{\pi}{2}$. **3**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a)



An open, flat topped water trough is in the shape of a triangular prism.

Its rectangular top measures 200 cm by 400 cm and its triangular cross-section has a vertical height of 25 cm.

When the water depth is h cm the water surface measures x cm by 400 cm.

(i) Show that when the water depth is h cm the volume V cm^3 of water in the trough is given by $V = 1600h^2$. **2**

Water is being emptied through a hole in its base at a constant rate of 16 L per second.

(ii) Find the rate at which the depth of water is changing when $h = 10$ cm. **2**

Question 5 continues on page 7

Marks

Question 5 (continued)

(b) After cooking her cheesecake, Donna puts it in the fridge. The fridge is running at a constant temperature of $8°$C. At time t minutes the temperature T of the cheesecake decreases according to the equation:

$$\frac{dT}{dt} = -k\,(T - 8) \text{ where } k \text{ is a positive constant.}$$

Donna puts the cheesecake in the fridge at 9.00am when its temperature is $85°$C.

(i) Show that $T = 8 + 77e^{-kt}$ satisfies both this equation and the initial conditions. **2**

(ii) Donna checks the temperature of the cheesecake at 10.00am and it is $40°$C. **3**

It is best served when it reaches a temperature of $10°$C.

At what time (to the nearest minute) should Donna serve the cheesecake?

(c) In the expansion of $(1 + ax)^{10}$, the coefficient of x^6 is twice the coefficient of x^7. **3**

Find the value of a.

End of Question 5

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) (i) In the expansion of $(a+b)^{20}$ show that $\dfrac{T_{n+1}}{T_n}$ is given by: **2**

$$\frac{21-n}{n}\ \frac{b}{a}$$

$\left(\text{where } T_{n+1} = {}^{20}C_n\ a^{20-n}\ b^n\right)$

(ii) In the game of craps, 2 dice are thrown and the score is recorded as the sum of the uppermost faces of the dice.

α) Find the probability that a score of 7 is recorded. **1**

β) If two dice are rolled 20 times, what is the most probable number of scores of 7 thrown? Calculate the probability that this occurs. **3**

(b) (i) Use the method of mathematical induction to show that if *x* is a positive integer then $(1+x)^n - 1$ is divisible by *x* for all positive integers $n \geq 1$. **3**

(ii) Factorise $12^n - 4^n - 3^n + 1$. **1**

(iii) Without using the method of mathematical induction, deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers $n \geq 1$. **2**

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$, the term independent of x is $(-1)^n \times {}^{2n}C_n$. **2**

(ii) Show that $\left(1 + x\right)^{2n}\left(1 - \frac{1}{x}\right)^{2n} = \left(x - \frac{1}{x}\right)^{2n}$ **1**

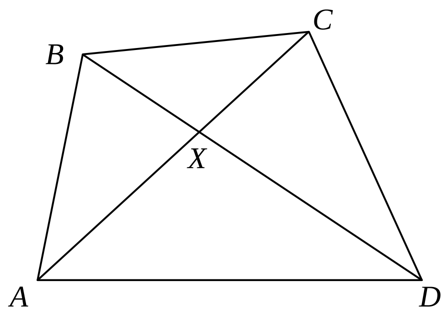
(iii) Deduce that:

$$\left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \left({}^{2n}C_2\right)^2 - \ldots + \left({}^{2n}C_{2n}\right)^2 = (-1)^n \times {}^{2n}C_n$$ **2**

Question 7 continues on page 10

Question 7 (continued)

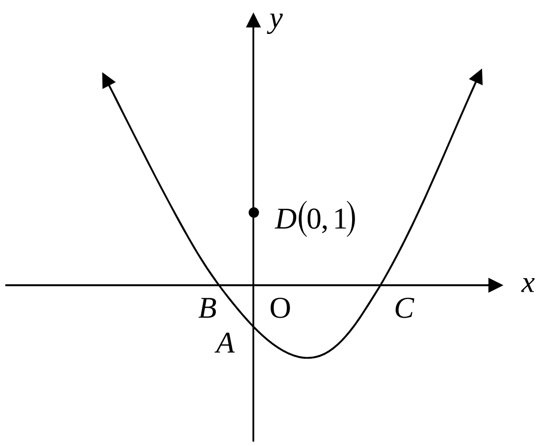
(b) (i) **4**



In the diagram above, the diagonals of quadrilateral *ABCD* intersect at *X*.

Show that if $AX \,.\, XC = BX \,.\, XD$, *ABCD* is a cyclic quadrilateral.

(ii)



Consider the parabola $y = x^2 + px - q$, where $q > 0$.
Let the parabola intercept the *y*-axis at *A* and the *x*-axis at the distinct points *B* and *C*.
D is the point $(0, 1)$

α) Find the co-ordinates of *B* and *C*. **1**

β) Show that *ABDC* is a cyclic quadrilateral. **2**

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Extension 1 Mathematics Trial HSC 2009 - Solutions

Q1 Reas - 3, Calc - 2

a) $\lim_{x\to 0} \frac{\sin 3x}{5x} = \frac{3}{5}\lim_{x\to 0}\frac{\sin 3x}{3x}$

$= \frac{3}{5}\times 1$ ✓

$= \frac{3}{5}$

Done well

b) $A(1,-4)$ $B(6,9)$

$-2:3$

$x = \frac{3\times 1 + -2\times 6}{-2+3}$ $y = \frac{3\times -4 + -2\times 9}{-2+3}$ ✓

$= \frac{-9}{1}$ $= \frac{-30}{1}$

$= -9$ $= -30$

$P(-9,-30)$ ✓

Those who used the 'cross' method were more successful than those who used formula.

c) $(x-1)^2 \frac{4}{x-1} \geq 1\,(x-1)^2$

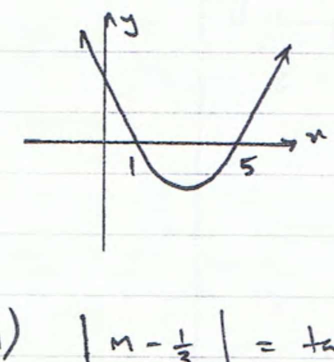
$4(x-1) \geq x^2-2x+1$

$4x-4 \geq x^2-2x+1$

$0 \geq x^2-6x+5$ ✓

Sketch $y = x^2-6x+5$

$= (x-5)(x-1)$



$1 < x \leq 5$ ✓ ✓

Reas 3

- Silly algebraic mistakes were made here. Take your time with Q1. Also $x \neq 1$

d) $\left|\frac{m-\frac{1}{3}}{1+m\times\frac{1}{3}}\right| = \tan\frac{\pi}{4}$

$\frac{|m-\frac{1}{3}|}{|1+\frac{m}{3}|} = 1$ ✓

$|m-\frac{1}{3}| = |1+\frac{m}{3}|$

You need to remember exact values of trig ratios. i.e. $\tan\frac{\pi}{4} = 1$ not $\frac{1}{\sqrt{2}}$

$m - \frac{1}{3} = 1 + \frac{m}{3}$ or $m - \frac{1}{3} = -1 - \frac{m}{3}$

$3m - 1 = 3 + m$ $\quad$ $3m - 1 = -3 - m$

$2m = 4$ $\quad$ $4m = -2$

$m = 2$ $\quad$ $m = -\frac{1}{2}$ ✓

e) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$ ✓

$= \frac{-3}{-5}$

$= \frac{3}{5}$ ✓

Done well

f) $\int \sec 2x \tan 2x \, dx = \frac{1}{2} \sec 2x + c$ ✓ ✓

Calc - 2

Done well

Q2 a) i) $\angle ACE = 36°$ (angle between tangent and chord equals angle in alternate segment) ✓

Reas 3

Comm 3

Calc 2

ii) $\angle CAE = 90°$ (angle in a semi-circle) ✓

$\angle FAC = 180 - 90 - 36$ (angle sum of a st. line)

$= 54°$

$\angle ADC = 54°$ (angle between tangent and chord equals angle in alternate segment) ✓

Comm - 3

b) $\int \frac{dx}{\sqrt{25 - 4x^2}} = \frac{1}{2} \int \frac{dx}{\sqrt{\frac{25}{4} - x^2}}$

$= \frac{1}{2} \sin^{-1}\left(\frac{2x}{5}\right) + c$ ✓ ✓

Calc 2

c) i) $\sin x - \sqrt{3}\cos x = R\sin x\cos\alpha - R\cos x\sin\alpha$

$\therefore R\sin\alpha = \sqrt{3}$ ...(1)

$R\cos\alpha = 1$ ...(2)

① ÷ ② $\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$ ✓

$①^2 + ②^2 = R^2\sin^2\alpha + R^2\cos^2\alpha = (\sqrt{3})^2 + 1^2$

$R^2 = 4$

$R = 2$ ✓ as $R > 0$

$\therefore \sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$

ii) $\sin x - \sqrt{3}\cos x = \sqrt{2}$

$2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{2}$

$\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$

$x - \frac{\pi}{3} = n\pi + (-1)^n \frac{\pi}{4}$ ✓ for $\frac{\pi}{4}$

$x = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{3}$ ✓

d) prob of colour blindness = 0.05

prob of not colour blindness = 0.95

i) $P(X=0) = {}^{20}C_0 (0.05)^0 (0.95)^{20}$

$= 0.358$ ✓

ii) $P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$ ✓

$= 1 - \left((0.95)^{20} + {}^{20}C_1 (0.05)(0.95)^{19}\right)$

$= 0.264$ ✓ Reas - 3

Q3 a) $\int_0^{3/2} \sqrt{9 - x^2}\, dx$ $\quad x = 3\sin\theta \quad dx = 3\cos\theta\, d\theta \quad x = \frac{3}{2}\ \ \theta = \frac{\pi}{6} \quad x = 0\ \ \theta = 0$

Reas - 3
Comm - 3
Calc - 5

$= \int_0^{\pi/6} \sqrt{9 - 9\sin^2\theta} \cdot 3\cos\theta\, d\theta$ ✓

$= \int_0^{\pi/6} 3\cos\theta \cdot 3\cos\theta\, d\theta$

$= 9\int_0^{\pi/6} \cos^2\theta\, d\theta$ $\quad \cos 2\theta = 2\cos^2\theta - 1 \quad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$= \frac{9}{2}\int_0^{\pi/6} 1 + \cos 2\theta\, d\theta$

This was done poorly by many students. This is a standard integral type so you need to revise the work.

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$$ ✓

$$= \frac{9}{2}\left[\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - (0+0)\right]$$

$$= \frac{9}{2}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$$

$$= \frac{3\pi}{4} + \frac{9\sqrt{3}}{8}$$ ✓ Calc - 3

b) i) No of triangles $= {}^{12}C_3$

$= 220$ ✓

Done well

ii) No of triangles $= {}^{7}C_1 \times {}^{4}C_1 \times {}^{1}C_1$

$= 28$

$\therefore$ Prob $= \frac{28}{220} = \frac{7}{55}$ ✓

Some people didn't read the question regarding probability.

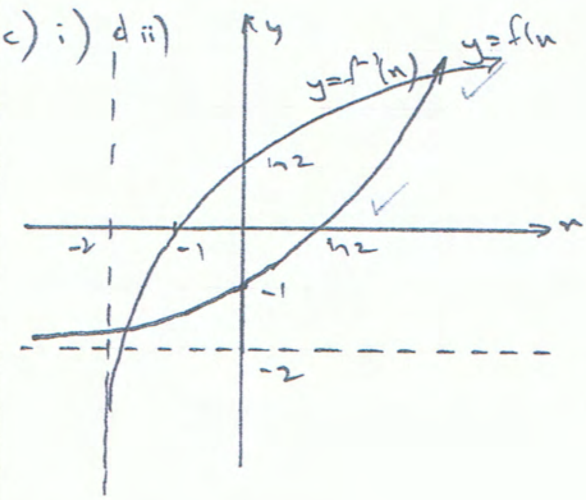
iii) No of triangles $= {}^{7}C_3 + {}^{4}C_3$

$= 35 + 4$

$= 39$

Prob $= \frac{39}{220}$ ✓ Reas - 3

c) i) & ii)



Comm - 1

Comm - 1

Done fairly well, although asymptotes were left out on many. Labelling x- and y intercepts would have been nice

iii) $f(x)$: $y = e^x - 2$

interchange x and y

$x = e^y - 2$

$x + 2 = e^y$

$y = \ln(x+2)$ ✓

Done well

iv) As the graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$, points of intersection lie on $y = x$

$\therefore$ they satisfy $e^x - 2 = x$ ✓

$\therefore e^x - x - 2 = 0.$ Comm-1

Done reasonably well
The key point is that $f(x)$ and $f^{-1}(x)$ intersect on $y=x$.

v) $f(x) = e^x - x - 2$

$f'(x) = e^x - 1$ ✓

$\therefore x_2 = 1.5 - \dfrac{f(1.5)}{f'(1.5)}$

$= 1.5 - \dfrac{e^{1.5} - 1.5 - 2}{e^{1.5} - 1}$

$= 1.22$ (to 2 dec. pl.) ✓ Calc 2

Done very well

Q4 a) i) $x^2 = 8y$

Reas - 5
Comm - 2
Calc - 3

$y = \dfrac{x^2}{8}$

$\dfrac{dy}{dx} = \dfrac{x}{4}$

$\therefore$ at P $m_{tang} = \dfrac{4p}{4}$ ✓

$= p.$

$\therefore m_{norm} = -\dfrac{1}{p}$

$\therefore y - 2p^2 = -\dfrac{1}{p}(x - 4p)$ ✓

$py - 2p^3 = -x + 4p$

$x + py = 2p^3 + 4p.$ Comm-2

ii) A: $x = 0$ $0 + py = 2p^3 + 4p$

$y = 2p^2 + 4$

$\therefore A(0, 2p^2 + 4)$ ✓ $P(4p, 2p^2)$

$\therefore$ midpt $\left(\dfrac{0 + 4p}{2}, \dfrac{2p^2 + 4 + 2p^2}{2}\right)$ ✓

$= (2p, 2p^2 + 2).$ Reas-2

iii) $x = 2p \quad y = 2p^2 + 2$

$p = \frac{x}{2}$ sub into y

$y = 2\left(\frac{x}{2}\right)^2 + 2$ ✓

$y = 2\frac{x^2}{4} + 2$

$4y = 2x^2 + 8$

$2y - 4 = x^2$

$x^2 = 2(y-2)$

$\therefore 4a = 2 \quad a = \frac{1}{2} \quad$ vertex $(0, 2)$ ✓

focus $(0, 2\frac{1}{2})$. ✓

Reas - 3

b) i) $a = 3$ ✓

$b = \frac{1}{2}$ ✓

ii) $y = 3\cos^{-1}\frac{x}{2}$

$\frac{y}{3} = \cos^{-1}\frac{x}{2}$

$\frac{x}{2} = \cos\frac{y}{3}$

$x = 2\cos\frac{y}{3}$

$\therefore A = \int_0^{\pi/2} 2\cos\frac{y}{3}\,dy$ ✓

$= 2\left[3\sin\frac{y}{3}\right]_0^{\pi/2}$ ✓

$= 6\left(\sin\frac{\pi}{6} - \sin 0\right)$

$= 6 \times \frac{1}{2}$

$= 3$ units^2. ✓

Calc - 3

Q5 a) i) $V = Ah$

Reas - 7

$= \frac{1}{2}xh \times 400$

Comm - 2

$= 200xh$

Calc - 3

now by similar triangles

$\frac{x}{h} = \frac{200}{25}$

$x = 8h$ ✓

$\therefore V = 200 \times 8h \times h$ ✓

$= 1600h^2$

Reas - 2

Similar triangle questions are quite popular so for those who didn't get this solution you need to do further practice.

ii) $\frac{dV}{dt} = -16\ L/s$ $\quad 1L = 1000cm^3$

$= -16000cm^3/s$

find $\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$\frac{dV}{dh} = 3200h$. ✓

$\therefore \frac{dh}{dt} = \frac{1}{3200h} \times -16000$ ✓

$= -\frac{5}{h}$.

$\therefore$ when $h=10$

$\frac{dh}{dt} = -\frac{5}{10}$

$= -\frac{1}{2}\ cm/s$. ✓ Calc - 3

Most students arrived at $\frac{dh}{dt} = \frac{dh}{dv} \times \frac{dv}{dt}$ but made a mistake with units i.e $16L = 16000cm^3$ You can't mix units in these questions

b) LHS $= \frac{dT}{dt}$

$= -k \times 77e^{-kt}$

RHS $= -k(T-8)$

$= -k(8 + 77e^{-kt} - 8)$

$= -k \times 77e^{-kt}$ ✓

$\therefore$ LHS = RHS

$\therefore T = 8 + 77e^{-kt}$ satisfies diff. eqn.

when $t=0$ $\quad T = 8 + 77e^{-k \times 0}$

$= 8 + 77$ ✓

$= 85°C$. Comm - 2

You need to be careful with show questions You need to have more than $\frac{dT}{dt} = -k77e^{-kt} = -k(T-8)$ initial conditions was also missed by many.

ii) $t = 60$ $\quad T = 40$

$\therefore 40 = 8 + 77e^{-k \times 60}$

$32 = 77e^{-k \times 60}$

$\frac{32}{77} = e^{-k \times 60}$

$-60k = \ln \frac{32}{77}$

$k = -\frac{\ln(32/77)}{60}$ ✓

Done very well.

find t when $T = 10$

$$10 = 8 + 77e^{-kt}$$
$$2 = 77e^{-kt}$$
$$\frac{2}{77} = e^{-kt}$$
$$-kt = \ln \tfrac{2}{77}$$
$$t = \frac{\ln\left(\frac{2}{77}\right)}{-k}$$ ✓

$= 249.45\ldots$ minutes

$= 4$hrs 9min

1:09 pm. ✓

Reas 3

c) coeff of x^6 : ${}^{10}C_6 a^6$

coeff of x^7 : ${}^{10}C_7 a^7$

$\therefore 210a^6 = 2 \times 120 a^7$ ✓

$a = \frac{210}{240}$

$= \frac{7}{8}$ ✓ Reas - 2

Most common mistake was $2 \times 210a^6 = 120a^7$

Q6 a) i) Reas - 8, Comm - 2

$$\frac{T_{n+1}}{T_n} = \frac{{}^{20}C_n a^{20-n} b^n}{{}^{20}C_{n-1} a^{20-(n-1)} b^{n-1}}$$

$$= \frac{\frac{20!}{(20-n)!\,n!} a^{20-n} b^n}{\frac{20!}{[20-(n-1)]!\,(n-1)!} a^{20-(n-1)} b^{n-1}}$$ ✓

$$= \frac{\frac{b}{n}}{\frac{a}{20-(n-1)}}$$ ✓ Comm - 2

$$= \frac{21-n}{n}\,\frac{b}{a}$$

ii) α) $P(7) = \frac{1}{6}$ ✓

β) $P(7) = {}^{20}C_n \left(\frac{5}{6}\right)^{20-n}\left(\frac{1}{6}\right)^n$

find n such that $\frac{T_{n+1}}{T_n} > 1$

$\therefore \frac{21-n}{n}\,\frac{\frac{1}{6}}{\frac{5}{6}} > 1$ from i). ✓

$$\frac{21-n}{n} \times \frac{1}{5} > 1$$

$$\frac{21-n}{5n} > 1$$

$21-n > 5n$ as $n > 0$

$6n < 21$

$n < 21/6$

$\therefore n = 3$ ✓

$\therefore$ the most likely number of 7's thrown is 3

$P(7) = {}^{20}C_3 \left(\frac{5}{6}\right)^{17}\left(\frac{1}{6}\right)^3$ ✓ Recs - 3

$= 0.238$ (to 3 d.p.)

b) i) show true for $n=1$

$(1+x)^1 - 1 = 1+x-1$

$= x$

which is divisible by x. ✓

assume true for $n=k$

$\therefore (1+x)^k - 1 = Mx$ where M is an integer

show true for $n=k+1$

i.e. $(1+x)^{k+1} - 1 = Qx$ where Q is an integer

LHS $= (1+x)^{k+1} - 1 = (1+x)(1+x)^k - 1$ ✓

$= (1+x)(Mx+1) - 1$ from above

$= Mx + \cancel{1} + Mx^2 + x \cancel{-1}$

$= x(M + Mx + 1)$ ✓

$= Qx$ as M and x are positive integers

$\therefore$ result is true for $n=k+1$ Recs - 3

$\therefore$ we have shown that if it is true for $n=k$ then the result is true for $n=k+1$. The result is true for $n=1$ $\therefore$ by the principle of Mathematical Induction the result is true for all positive n.

ii) $12^n - 4^n - 3^n + 1 = 3^n \times 4^n - 4^n - 3^n + 1$
$= 4^n(3^n - 1) - 1(3^n - 1)$
$= (3^n - 1)(4^n - 1)$ ✓

iii) $12^n - 4^n - 3^n + 1 = (3^n - 1)(4^n - 1)$
$= ((2+1)^n - 1)((3+1)^n - 1)$ ✓

now $(2+1)^n - 1$ is divisible by 2 from i)
& $(3+1)^n - 1$ is divisible by 3 from i) ✓
$\therefore (3^n - 1)(4^n - 1)$ is divisible by $2 \times 3 = 6$.
Re - 2

Res 7
Comm 5

Q7 a) i) $(x - \frac{1}{x})^{2n} = \sum_{r=0}^{2n} {}^{2n}C_r (x)^{2n-r} (-\frac{1}{x})^r$

Done fairly well

$= \sum_{r=0}^{2n} {}^{2n}C_r (x)^{2n-r} (-x)^{-r}$ ✓

$\therefore$ for the term independent of x:
$2n - r - r = 0$ ✓
$2r = 2n$
$r = n$.
$\therefore$ term is ${}^{2n}C_n (x)^{2n-n} (-x)^{-n}$
$= {}^{2n}C_n (-1)^n$ Res - 2

ii) $LHS = (1+x)^{2n}(1 - \frac{1}{x})^{2n} = [(1+x)(1 - \frac{1}{x})]^{2n}$
$= (1 - \frac{1}{x} + x - 1)^{2n}$ ✓
$= (x - \frac{1}{x})^{2n}$ Comm - 1
$= RHS.$

Easy mark thrown away by many. Just because this question is complicated there are still easy parts

iii) expand $(1+x)^{2n}(1 - \frac{1}{x})^{2n}$
$= ({}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \dots + {}^{2n}C_{2n} x^{2n})$
$\times ({}^{2n}C_0 - {}^{2n}C_1 (\frac{1}{x}) + {}^{2n}C_2 (\frac{1}{x^2}) - \dots + {}^{2n}C_{2n} \frac{1}{x^{2n}})$

~~Few the~~ Many students didn't know which terms to multiply together.

$= \dots + {}^{2n}C_0 \times {}^{2n}C_0 - {}^{2n}C_1 x \, {}^{2n}C_1 (\frac{1}{x}) + \dots + {}^{2n}C_{2n} x^{2n} \times {}^{2n}C_{2n} \frac{1}{x^{2n}} + \dots$ ✓

$\therefore$ terms independent of x in this expansion are:
$({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \dots + ({}^{2n}C_{2n})^2$

now $(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n} = \left(x-\frac{1}{x}\right)^{2n}$

and the term independent of x in $\left(x-\frac{1}{x}\right)^{2n}$

is $(-1)^n \, {}^{2n}C_n$ ✓

Reas - 2

∴ equating terms

$$\left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \left({}^{2n}C_2\right)^2 - \dots + \left({}^{2n}C_{2n}\right)^2 = (-1)^n \, {}^{2n}C_n$$

b) i) In ΔAXB and ΔDXC

$\frac{AX}{DX} = \frac{XB}{XC}$ (as $AX \cdot XC = BX \cdot XD$) ✓

$\angle BXA = \angle CXD$ (vert. opp angles are eq.) ✓

∴ $\Delta AXB \parallel\parallel \Delta DXC$ (two pairs of sides in the same ratio and the included angle is equal) ✓

∴ $\angle ABX = \angle DCX$ (corr. angles in similar triangles)

∴ ABCD is a cyclic quadrilateral as angles in the same segment are equal. ✓

Comm - 4

Most students who attempted this question knew the concepts involved but many struggled with presenting a formal and logical proof.

ii) α) B and C are x-intercepts ∴ $y=0$

$$\therefore x^2 + px - q = 0$$

$$x = \frac{-p \pm \sqrt{p^2 - 4.1.-q}}{2\times 1}$$

$$= \frac{-p \pm \sqrt{p^2 + 4q}}{2}$$

$$\therefore B\left(\frac{-p-\sqrt{p^2+4q}}{2}, 0\right) \text{ \& } C\left(\frac{-p+\sqrt{p^2+4q}}{2}, 0\right)$$ ✓

$$\therefore OB = \left|\frac{-p-\sqrt{p^2+4q}}{2}\right| \qquad OC = \left|\frac{-p+\sqrt{p^2+4q}}{2}\right|$$

$$= \frac{p+\sqrt{p^2+4q}}{2} \qquad = \frac{-p+\sqrt{p^2+4q}}{2}$$

$$\therefore OB \times OC = \frac{p+\sqrt{p^2+4q}}{2} \times \frac{-p+\sqrt{p^2+4q}}{2}$$

$$= \frac{-p^2 + p\sqrt{p^2+4q} - p\sqrt{p^2+4q} + \left(\sqrt{p^2+4q}\right)^2}{4}$$

Done well for those students who knew what to do!

Be careful that OB and OC are lengths so $OB = \left|\frac{-p-\sqrt{p^2+4q}}{2}\right|$

$$= \frac{-p^2 + p^2 + 4q}{4}$$

$$= q$$

$OA = |q|$ $\quad$ $OD = |1|$

$= q$ $\quad$ $= 1$

$\therefore OA \times OD = q \times 1$

$= q$

$\therefore OB \times OC = OA \times OD$

$\therefore$ ABDC is a cyclic quad.